A new lower bound for the minimal singular value for real non-singular matrices by means of matrix trace and determinant

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Abstract

We present a new lower bound on minimal singular values of real matrices base on Frobenius norm and determinant. We show, that under certain assumptions on matrix A is our estimate sharper than two recent ones based on a matrix norm and determinant.

1. Introduction

The eigenvalues or singular values of real matrices are difficult to evaluate in general. It is however useful to know an approximate location of these values. For a Hermitian positive definite matrix, the ratio of the largest to the smallest eigenvalue is useful in determining whether the equation $Ax = b$ is ill-conditioned or not. Probably the first bounds for eigenvalues have been achieved already more than a hundred years ago. Possibly the best-known inequality on eigenvalues is from Gerschgorin in 1931 [1]. The first paper using traces in eigenvalue inequalities was from Schur in 1909 [3]. Our paper deals with lower bounds on the minimal singular values.

2. Preliminaries and recent estimates

Let A be a $n \times n$, $n \geq 2$ matrix with real elements. Let $\|A\|_E = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2}$ be the Frobenius (or Euclidean) norm of matrix A. Trace of a $n \times n$ matrix A denotes $tr(A) = \sum_{i=1}^{n} a_{ii}$. The spectral norm of the matrix A is $\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i}$, where λ_i is eigenvalue of $A^T A$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix A, then $detA = \lambda_1 \lambda_2 \ldots \lambda_n$. Denote the smallest singular value of A by σ_n and its largest singular value by σ_1. It holds that $\|A\|_E^2 = \sum_{i=1}^{n} \sigma_i^2 = tr(A^T A)$, where trace $tr(A^T A) = \sum_{i=1}^{n} \sigma_i^2$. The condition number of matrix A is $\kappa(A) = \frac{\sigma_{max}}{\sigma_{min}}$.

Yu Yi-Sheng and Gu Dun-he gave in 1997 [6] a lower bound for σ_n for a nonsingular matrix as

$$\sigma_{min} \geq (\frac{n-1}{\|A\|_E^2})^{(n-1)/2}|detA|, \tag{1}$$

and G. Piazza and T. Politi gave in 2002 [4] a lower bound on the minimal singular matrix with positive singular values as

$$\sigma_{min} \geq \frac{|detA|}{2^{(n-2)/2}\|A\|_E}. \tag{2}$$

In 2007 Turkmen and Civcic in [7] also used matrix norm and determinant for finding upper bounds for maximal and minimal singular value of positive definite matrices.
For symmetric positive definite matrix A, we can suppose $\|A\|_2 = 1$, i.e. that the matrix A is normalized, where $\|.\|_2$ is the Euclidean norm. Consequently for its condition number is $\kappa(A) = \|A\|_2\|A^{-1}\|_2 = \frac{1}{\sigma_n}$. The matrix normalization can be always achieved by multiplying the set of equations $Ax = b$ by a suitable constant or for example by the divisive normalization defined by Weiss [8] or Ng et al. [5]), which uses the Laplacian L of the symmetric positive definite matrix A. The transformation is defined by $D^{-1/2}AD^{-1/2}$, where $D = \{d_{ij}\}_{i,j=1}^n$ and $d_{ij} = 0$ for $i \neq j$ and $d_{ij} = \sum_{j=1}^n a_{ij}$ for $i = j$, where $A = \{a_{ij}\}_{i,j=1}^n$.

3. Our result

Theorem 1. *Let A be a nonsingular matrix with singular values σ_i so that $|\sigma_{max}| = |\sigma_1| \geq \ldots \geq |\sigma_n| = |\sigma_{min}|$ and let $|\sigma_{max}| \neq |\sigma_{min}|$. Let $\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$ be the Euclidean norm of matrix A.*

(i) Then for its minimal and maximal singular values holds

$$0 < \left(\frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|detA|^{2/n}})}\right)^{1/2} \leq \sigma_{min}. \tag{3}$$

(ii) For $|\sigma_{max}| = 1$ (supposing that $|detA| \neq 1$) holds

$$0 < \left(\frac{|detA|^{2/n}(\|A\|^2 - 1)}{n(|detA|^{2/n} - 1)}\right)^{1/2} \leq \sigma_{min}.$$

Proof:
(i) We will apply the following result of Diaz and Metcalf ([2]) which is a stronger form of Pólya-Szegö and Kantorovich's inequality. Let the real numbers $a_k \neq 0$ and b_k $(k = 1, \ldots, n)$ satisfy $m \leq \frac{b_k}{a_k} \leq M$. Then $\sum_{k=1}^m b_k^2 + mM\sum_{k=1}^n a_k^2 \leq (m + M)\sum_{k=1}^n a_k b_k$.

Let $b_k = \sigma_k$, $a_k = \frac{1}{\sigma_k}$, $m = \sigma^2_{min}$, $M = \sigma^2_{max}$, let $m \neq M$. Then from the Diaz and Metcalf's inequality follows, that $\sum \sigma_k^2 + mM \sum \frac{1}{\sigma_k^2} \leq (M+m)m$, which is equivalent to $\|A\|_E^2 \leq Mn + mn - \frac{mMn}{|detA|^{2/n}}$ and that to $\frac{\|A\|_E^2}{n} - M \leq m(1 - \frac{M}{|detA|^{2/n}})$ and from that follows $\frac{\|A\|_E^2 - Mn}{n(1-\frac{M}{|detA|^{2/n}})} \leq m = \sigma^2_{min}$ and the statement of theorem follows.
1.
(i) It holds that $0 < \frac{\|A\|_E^2 - Mn}{n(1-\frac{M}{|detA|^{2/n}})}$, since to be true must hold ($\|A\|_E^2 - Mn > 0$ and $1 - \frac{M}{|detA|^{2/n}} > 0$) or ($\|A\|_E^2 - Mn < 0$ and $1 - \frac{M}{|detA|^{2/n}} < 0$). The second term $\|A\|_E^2 - Mn < 0$ and $1 - \frac{M}{|detA|^{2/n}} < 0$ holds, since $\|A\|_E^2 = \sum_{i=1}^n \sigma_i^2 < \sigma_{max} n$ and $\Pi_{i=1}^n \sigma_i^2 = |detA|^2 < M^n = (\sigma^2_{max})^n$. (ii) follows form (i).

4. Comparison with other lower estimates

Theorem 2. *Let for a $n \times n$- nonsingular matrix A with $n > 2$ holds $|detA| \neq 1$. Then*

$$0 \leq \frac{|detA|}{2^{\frac{n-2}{2}} \|A\|_E} < \left(\frac{\|A\|_E^2 - n\sigma^2_{max}}{n(1 - \frac{\sigma^2_{max}}{|detA|^{2/n}})} \right)^{1/2} \leq \sigma_{min}, \tag{4}$$

respectively for $|\sigma_{max}| = 1$

$$0 \leq \frac{|detA|}{2^{\frac{n-2}{2}} \|A\|_E} < \left(\frac{\|A\|_E^2 - n}{n(1 - \frac{1}{|detA|^{2/n}})} \right)^{1/2} \leq \sigma_{min}, \tag{5}$$

i.e. the lower bound from Theorem 1 is sharper than the lower bound (2) from G. Piazza and T. Politi.

Proof:
We prove first the inequation (5). Assume that the opposite of (5) holds, i.e.

$$\frac{|detA|^2}{2^{n-2} \|A\|_E^2} \geq \frac{\|A\|_E^2 - n}{n(1 - \frac{1}{|detA|^{2/n}})} \tag{6}$$

This is equivalent to

$$\frac{|detA|^2}{2^{n-2} \|A\|_E^2} \geq \frac{\frac{1}{n}\sum \sigma_i^2 - 1}{1 - \frac{1}{(\Pi \sigma_i^2)^{1/n}}}. \tag{7}$$

One can easily observe that

$$\frac{|detA|}{\|A\|_E} = \sqrt{\frac{\Pi \sigma_i^2}{\sum \sigma_i^2}} \leq 1; \tag{8}$$

Applying the relationship of geometric and arithmetic mean, for the right hand side of (7) holds

$$\frac{(\frac{1}{n}\sum \sigma_i^2 - 1)(\Pi\sigma_i^2)^{1/n}}{(\Pi\sigma_i^2)^{1/n} - 1} \geq (\Pi\sigma_i^2)^{1/n}.$$

Then together with (8) and the left hand side of (7) is

$$\frac{1}{2^{n-2}} \geq (\Pi\sigma_i^2)^{1/n}.$$

$$(\Pi\sigma_i^2)^{-1/n} \geq 2^{n-2}.$$

$$\frac{-1}{n}\sum \lg_2 \sigma_i^2 \geq n-2$$

$$0 > -\sum \lg_2 \sigma_i^2 \geq n(n-2),$$

which leads for every $n > 2$ to a contradiction, therefore the statement (5) holds. To prove (4), by the analogical proof by contradiction we get

$$\frac{|detA|^2}{2^{n-2}\|A\|_E^2} \geq \frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|detA|^{2/n}})} = \frac{(\frac{1}{n}\sum \sigma_i^2 - \sigma_{max}^2)(\Pi\sigma_i^2)^{1/n}}{(\Pi\sigma_i^2)^{1/n} - \sigma_{max}^2} \geq (\Pi\sigma_i^2)^{1/n}. \quad (9)$$

and the rest of the proof is identical to the previous case.

We will show that our estimate of minimal singular value is under certain conditions also sharper than the estimate from Yu Yi-Sheng and Gu Dun-he.

Theorem 3. *Let for a $n \times n$- nonsingular matrix A with $n > 1$.*

(i) If holds $|detA| > 1$ then

$$\frac{(n-1)^{n-1}}{\|A\|^{2n-2}(\|A\|^2 - n)} \geq \frac{1}{n|detA|^2(1 - \frac{1}{|detA|}^{2/n})}, \quad (10)$$

i.e. the estimate from Yu Yi-Sheng and Gu Dun-he is sharper than ours.

(ii) If for matrix A holds $|detA| < \min\{1, \frac{\|A\|}{(n-1)^{n/2}}\}$ then

$$\frac{(n-1)^{n-1}}{\|A\|^{2n-2}(\|A\|^2 - n)} < \frac{1}{n|detA|^2(1 - \frac{1}{|detA|}^{2/n})}, \quad (11)$$

i.e. our estimate is sharper than the estimate (1) from Yu Yi-Sheng and Gu Dun-he.

Proof: (i) Can be easily seen from the fact, that $\frac{1}{|detA|^2\|A\|^{\frac{2n-2}{n}}} \leq 1$.
(ii): Assume that $|detA| < 1$. The inequality (11) is equivalent to

$$\frac{\|A\|^{2n-2}|detA|^{\frac{2}{n}}(\|A\|^2 - n)}{|detA|^2(|detA|^{\frac{2}{n}} - 1)} > n(n-1)^{n-1} \quad (12)$$

Since between arithmetical mean A and geometrical mean G holds $A \geq G$, in our case is $\frac{1}{n}\|A\|^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 =\geq (\Pi_{i=1}^{n}\sigma_i^2)^{\frac{1}{n}} = |detA|^{\frac{2}{n}}$ and consequently $\frac{\frac{1}{n}\|A\|^2-1}{|detA|^{\frac{2}{n}}-1} \leq 1$. Then (12) can be rewritten as

$$\|A\|^{2n-2}|detA|^{\frac{2-2n}{n}}n > n(n-1)^{n-1}$$

and consequently

$$\|A\|^{2n-2}|detA|^{\frac{2-2n}{n}} > (n-1)^{n-1},$$

which corresponds to the initial assumption

$$[\frac{\|A\|}{|detA|^{\frac{1}{n}}}]^{2n-2} > (\sqrt{n-1})^{2n-2}, \tag{13}$$

and the statement is proven.

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